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Robust methods for stabilization of Hamiltonian systems in economic growth models ⋆

Alexander Tarasyev *,**, **Anastasiia Usova** *,***

* *Krasovskii Institute of Mathematics and Mechanics, Ural Branch, Russian Academy of Sciences, S. Kovalevskaya str. 16, 620990 Ekaterinburg, Russia (e-mail: tam@imm.uran.ru, Anastasy.Ousova@gmail.com)*
** *Ural Federal University, Mira str., 19, 620002 Ekaterinburg, Russia*
*** *University of Western Ontario, 1151 Richmond Street London, Ontario, Canada, N6A 3K7*

Abstract: The paper discusses the existence of a linear manifold in a vicinity of a steady state for stabilization of the Hamiltonian systems arising in optimal control problems for economic growth models. It is shown that such stable manifold exists for almost all possible values of model parameters guaranteeing the existence of a steady state. Research is based on the qualitative analysis of the Hamiltonian dynamics, which plays a key role for investigating the asymptotic behavior of optimal trajectories. A procedure is proposed for stabilization of the Hamiltonian system, whose trajectories converge to equilibrium and approximate the optimal solution with the quadratic accuracy at a vicinity of the steady state. Basing on properties of the Hamiltonian matrices, the classification of steady states is provided and the sensitivity analysis for identification of their character is implemented with respect to model parameters. The proposed approach is applied to the model dealing with dynamic optimization of the resource productivity.

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1. INTRODUCTION

The paper is devoted to analysis of the Hamiltonian systems arising in optimal control problems with infinite time horizon. These problems are based on economic growth models and operate with a number of factors for constructing the forecasts of economic development of a country or region (see Ayres and Warr (2009); Grossman and Helpman (1991); Krasovskii and Tarasyev (2007); Tarasyev and Usova (2010)).

Starting from a quite general structure of the model, the optimal control problem is posed for the integral payoff functional optimized on the model trajectories. Analysis of the problem is based on the Pontryagin maximum principle for problems with infinite horizon (see Pontryagin et al. (1962); Aseev and Kryazhimskiy (2007)). First, it is shown that in the considered problem the Hamiltonian function is strictly concave with respect to control parameters. Next, we construct the Hamiltonian systems and, assuming that a steady state exists, describe the Jacobian matrix at equilibrium and investigate its properties. We consider the linearized Hamiltonian system in a vicinity of the steady state and try to find such its solution, that belongs to the stable manifold Ω, where conjugate variables can be linearly expressed through the phase vector (see Ledyaev (2011)).

Using properties of the Jacobian (Tarasyev and Usova, 2015) and the Hamiltonian matrices, it is shown that for almost all values of the model parameters the linearized Hamiltonian system has a steady state of the stable character. This fact allows to find a stable manifold Ω and stabilize the original Hamiltonian system in a vicinity of the steady state.

Basing on this methodology, we generate solutions for the Hamiltonian system arising in the resource productivity model, and show that for different values of model parameters the proposed method ensures the existence of stable solutions which approximate optimal trajectories of the control problem.

2. OPTIMAL CONTROL PROBLEM

Let $x = (x_1, x_2, \dots, x_n)$, $x_i > 0, i = 1, 2, \dots, n$ be a vector of production factors whose impact on output y is described by the production function $y = f(x)$. The production function $y = f(x)$ has the following properties
P1. It is a monotone increasing function, *i.e.*
$$\partial f(x)/\partial x_i > 0, \quad \forall\, x_i > 0,\ i = 1, 2, \dots, n,$$
P2. It is strictly concave, *i.e.* the Hessian $H_f(x)$ is negative definite for all vectors x with positive coordinates.

Dynamics of production factors is described by the following system of differential equations

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$$\dot{x}(t) = F(x(t))u(t) + G(x(t)) = \Phi(x(t), u(t)), \quad (1)$$
$$x(0) = x^0. \quad (2)$$

Here symbols $F(\cdot)$, $G(\cdot)$ denote the functional matrix $F(\cdot) = \{f_{ij}(\cdot)\}_{i,j=1}^{n,m}$, and the vector–function $G(\cdot) = \{g_i(\cdot)\}_{i=1}^{n}$, respectively. The symbol $u(\cdot) = (u_1(\cdot), \dots, u_m(\cdot))$ stands for the control parameter of investments. Functions $f_{ij}(\cdot)$ and $g_i(\cdot)$ are twice continuously differentiable ($i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$).

Within the assumption on closedness of the economy system it is supposed that the output $y(t)$ can be shared between investments $u_i(t)y(t)$, $u_i(t) \geq 0$, $i = 1, \dots, m$ in improving productivity of production factors and consumption $c(t)$

$$y(t) = c(t) + \sum_{i=1}^{m} (u_i(t) + w_i(x(t)))y(t). \quad (3)$$

This equality provides restrictions on control parameters

$$0 < \sum_{i=1}^{m} u_i(t) < 1 \;\Rightarrow\; \exists\, \bar{u}_i \in (0, 1): \; u_i(t) \in [0, \bar{u}_i], \quad (4)$$

and determines the consumption level $c(t) = y(t)(1 - \sum_{i=1}^{m} (u_i(t) + w_i(x(t))))$. It is assumed that the consumption $c(t)$ can be approximated by the expression

$$c(t) \approx \prod_{i=1}^{m} (1 - u_i(t) - w_i(x(t)))y(t). \quad (5)$$

The utility function of the control process is defined as an integral consumption index discounted on the infinite time horizon with the positive discount factor ρ

$$J(\cdot) = \int_0^{+\infty} e^{-\rho t} \ln c(t) dt. \quad (6)$$

Problem 1. The problem is to find such control process $(x^0(t), u^0(t))$ which maximizes the utility function (6) along trajectories of the system (1)–(2) under the control restriction (4).

3. PROBLEM ANALYSIS

Problem analysis is implemented within the framework of the Pontryagin maximum principle extended to control problems with infinite time horizon (see Pontryagin et al. (1962), Aseev and Kryazhimskiy (2007)).

Let $H(x, \psi, u)$ be the stationary Hamiltonian function of the posed control problem [1]

$$H(\cdot) = \sum_{i=1}^{m} \ln (1 - u_i - w_i(x)) + \ln f(x) + \psi^T \Phi(x, u). \quad (7)$$

Lemma 2. The Hamiltonian function $H(x, \psi, u)$ (7) is strictly concave with respect to the control parameters u.

Proof. Evaluating the second derivative of the Hamiltonian function (7) with respect to control u, one can obtain the following Hessian

$$\partial H(\cdot, u)/\partial u_j = - (1 - u_j - w_j(x))^{-1} + \psi^T F_j(x), \quad (8)$$
$$\frac{\partial^2 H(x, \psi, u)}{\partial u_j \partial u_k} = \begin{cases} - (1 - u_j - w_j(x))^{-2}, & k = j \\ 0, & k \neq j. \end{cases},$$

[1] Symbol $(\cdot)^T$ means transposition

where $F_j(x) = \left(f_{1j}(x), f_{2j}(x), \dots f_{nj}(x)\right)^T$, $j = 1, \dots, m$. Hence, the Hessian is a diagonal matrix with negative diagonal elements. It proves the concavity of the Hamiltonian function (7) with respect to variable u. ∎

Using equality (8), one can evaluate values of control vector u^0 providing maximum to the Hamiltonian (7)

$$u_j^0(x, \psi) = \begin{cases} 0, & (x, \psi) \in \Delta_j^1 \\ 1 - w_j(x) - \Gamma_j(x, \psi), & (x, \psi) \in \Delta_j^2 \\ \bar{u}_j, & (x, \psi) \in \Delta_j^3 \end{cases} \quad (9)$$

$$\Delta_j^1 = \{(x, \psi) \colon w_j(x) + \Gamma_j(x, \psi) \geq 1\},$$
$$\Delta_j^2 = \{(x, \psi) \colon 1 - \bar{u}_j \leq w_j(x) + \Gamma_j(x, \psi) \leq 1\},$$
$$\Delta_j^3 = \{(x, \psi) \colon w_j(x) + \Gamma_j(x, \psi) \leq 1 - \bar{u}_j\},$$

where $\Gamma_j(x, \psi) = \left(\psi^T F_j(x)\right)^{-1}$, ($j = 1, \dots, m$). There exist 3^m domains with different control regimes. The domain $\mathcal{D} = \bigcap_{j=1}^{m} \Delta_j^2$, where all variables $u_j^0(x, \psi)$ are not constant, is called *domain of transition control*.

One can verify that the maximized Hamiltonian function

$$H_0(x, \psi) = H(x, \psi, u^0). \quad (10)$$

is continuous and smooth in variables x and ψ in all domains corresponding to different control regimes. This fact can be proved by using properties of functions generating the maximized Hamiltonian.

The Hamiltonian system is constructed by the formulae

$$\dot{x}(t) = \partial H_0(x, \psi)/\partial \psi, \quad \dot{\psi}(t) = \rho\psi - \partial H_0(x, \psi)/\partial x. \quad (11)$$

Next assumptions relate to the existence of a steady state, its properties and location.

A1. Let $P^* = (x^*, \psi^*)$ be a steady state of the Hamiltonian system (11), and $P^* \in \mathcal{D}$.

A2. The phase coordinates x^* of the steady state P^* are positive numbers, and the coordinates ψ^* corresponding to the conjugate variables ψ are non-zero.

4. STABLE MANIFOLD

Let us construct a stable manifold of the Hamiltonian in a neighborhood O_δ^* of the steady state P^*. First of all, it is necessary to find the Jacobian matrix of the system (11) evaluated at the steady state P^*.

$$J^* = \begin{pmatrix} A & B \\ C & \rho\mathbb{E}_n - A^T \end{pmatrix}, \quad \text{where} \quad (12)$$

$$A = \frac{\partial^2 H_0(P^*)}{\partial \psi \partial x}, \; B = \frac{\partial^2 H_0(P^*)}{\partial \psi^2}, \; C = -\frac{\partial^2 H_0(P^*)}{\partial x^2} \quad (13)$$

Consider a linearized Hamiltonian system in a vicinity O_δ^* of the steady state

$$\begin{cases} \dot{x} = A\widetilde{x} + B\widetilde{\psi}, & \widetilde{x}(t) = x(t) - x^* \\ \dot{\psi} = C\widetilde{x} + \left(\rho\mathbb{E}_n - A^T\right)\widetilde{\psi}, & \widetilde{\psi}(t) = \psi(t) - \psi^* \end{cases} \quad (14)$$

Construction of a stable manifold Ω is reduced to the following problem.

Problem 3. The problem is to find a matrix $X \in \mathbb{R}^{n \times n}$ such that the linear subspace of solutions of (14) satisfying

$$\widetilde{\psi}(t) = X\widetilde{x}(t) \quad (15)$$

is nonempty for any initial condition $\widetilde{x}_0 = x_0^* - x^*$ such that the point $(x_0^*, \psi^* + X\widetilde{x}_0)$ belongs to O_δ^*.

The linearized system (14) in variables $\xi = \widetilde{x}\ e^{-\rho/2t}$, $z = \widetilde{\psi}\ e^{-\rho/2t}$ has the form

$$\begin{cases} \dot{\xi} = (A - \rho/2\mathbb{E}_n)\,\xi + Bz, \quad \xi_0 = x_0^* \\ \dot{z} = C\xi - \left(A^T - \rho/2\mathbb{E}_n\right) z,\ z_0 = \psi^* + X\widetilde{x}_0 \end{cases} \tag{16}$$

Matrix M of the linear system (16)

$$M = \begin{pmatrix} A - \rho/2\mathbb{E}_n & B \\ C & \rho/2\mathbb{E}_n - A^T \end{pmatrix}. \tag{17}$$

is a Hamiltonian matrix, whose spectrum is symmetrical with respect to the imaginary axis (Paige and Loan, 1981). Here we assume that matrix M does not have pure imaginary eigenvalues. It implies that matrix M has exactly n eigenvalues with negative real part, and n others with positive real parts. Let $V = [v_1, \ldots, v_n]$ be eigenvectors corresponding to eigenvalues of matrix M with negative real parts.

Remark 4. Due to the fact that matrix M is a real-valued matrix, then for a complex eigenvalue μ, its complex-conjugate $\overline{\mu}$ is an eigenvalue of M also. Moreover, if $Mv = \mu v$ then $M\overline{v} = \overline{\mu v}$. Thus, if vector v_0 in V has a complex-valued component, then V contains vector $\overline{v}_0$. Therefore, instead of complex-valued vectors v_0, $\overline{v}_0$, one can consider their linear combination

$$v_1 = 0.5 \cdot (v_0 + \overline{v}_0)\,, \quad v_2 = 0.5 \cdot \mathrm{Im}\,(v_0 - \overline{v}_0)\,. \tag{18}$$

Next, let us suppose that all pairs of the complex conjugate eigenvectors in V are replaced by their linear combinations, as it is done in (18).

Desired *stable manifold* for the system (16), we determine as subspace Ω generated by the vectors $(v_1, \ldots, v_n)$

$$\Omega = \mathrm{span}\{v_1, \ldots, v_n\}. \tag{19}$$

Any vector $(\xi, \varphi)^T \in \Omega$ can be represented by the formula

$$\begin{pmatrix} \xi \\ z \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \nu, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix},\ V_1, V_2 \in \mathbb{R}^{n\times n}. \tag{20}$$

Assume that matrix V_1 is nonsingular. Hence, one can express the conjugate variable z through the phase vector ξ as follows $\nu = V_1^{-1}\xi \ \Rightarrow\ z = V_2V_1^{-1}\xi$. Finally, we get the desired matrix X in the form

$$X = V_2V_1^{-1}. \tag{21}$$

The obtained matrix provides the solution for the problem 3, since $z = X\xi \ \Rightarrow\ ze^{\rho/2t} = X\xi e^{\rho/2t} \ \Rightarrow\ \widetilde{\psi} = X\widetilde{x}$.

Thus, the following theorem is proved.

Theorem 5. Let system (14) have an asymptotically stable solution satisfying (15), then the stable manifold Ω (19) is constructed by eigenvectors corresponding to negative eigenvalues of matrix M, and the desired matrix X satisfies equality (21).

In should be mentioned here that the similar problem is considered in the paper Ledyaev (2011), where an analytical representation of matrix X is proposed through the solution of algebraic or differential Riccati equations.

Using the obtained relation (15) between phase and adjoint variables, one can construct a nonlinear stabilizer for the system (11) in the form

$$\widehat{u}(x) = u^0(x, \widehat{\psi}(x)), \quad \widehat{\psi}(x) = \psi^* + X(x - x^*), \tag{22}$$

where $u^0(\cdot,\cdot)$ is an optimal control determined in (9). The control law $\widehat{u}(x)$ (22) generates the following stable system of differential equations in the subspace Ω (19) with respect to phase variables only

$$\dot{x}(t) = F(x)\widehat{u}(x) + G(x). \tag{23}$$

The stabilized system (23) has the stable equilibrium at the point x^*, and its linearized system has the form

$$\dot{x} = (A + BX)\,(x - x^*). \tag{24}$$

Eigenvalues of the matrix $(A + BX)$ are $\lambda_1, \ldots, \lambda_n$ (moreover, $\lambda_i = \mu_i^- + \rho/2$ and $\mathrm{Re}\,(\lambda_i) < 0$, where μ_i^- is the eigenvalue of the matrix M with the negative real part, $i = 1, \ldots, n$) and columns of the sub-matrix V_1 (20) are eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$.

5. APPLICATION TO OPTIMIZATION MODEL OF RESOURCE PRODUCTIVITY

For illustration of the presented results, let us consider the dynamic optimization model of the resource productivity, which is investigated in (Tarasyev and Usova, 2016).

5.1 Optimization of resource productivity

In this section , we discuss the model of optimization of the resource productivity and formulate the corresponding optimal control problem. A detailed model description can be found in (Tarasyev and Zhu, 2012).

The model of the effective resource consumption (Tarasyev and Zhu, 2012) is based on the assumption that the resource stock is exhaustible, while the growth of the resource productivity can be regulated by investing in the technological sector of the economy.

The cumulative resource consumption $M(t)$ is the summarized amount of the resource use $R = R(t)$, which satisfies the condition of the resource limitation

$$M(t) = \int_0^t R(s)ds \leqslant M_0. \tag{25}$$

Production $y(t)$ depends on capital stock $K(t)$ and resource use $R(t)$, and this dependence is described by the production function of the Cobb–Douglas type

$$y(t) = ae^{bt}R^\alpha(t)K^{1-\alpha}(t),\ a > 0,\ b \geqslant 0,\ \alpha \in (0, 1). \tag{26}$$

A positive coefficient a is the scale factor; a nonnegative rate b describes the growth process of production $y(t)$ due to development of basic production factors, such as technology, labor, etc. Parameter α denotes the elasticity coefficient of the natural resource use. The elasticity coefficient of the capital stock is equal to $(1 - \alpha)$.

Assumption 6. It is assumed that capital stock $K(t)$ is a more effective production factor in comparison with resource use $R(t)$. To emphasize significance of capital $K(t)$ with respect to resource use $R(t)$, it is assumed that the following inequality is valid for elasticity coefficients $1 - \alpha > \alpha$, i.e. $\alpha < 0.5$.

By the symbol $Z(t)$ we denote the resource productivity

$$Z(t) = y(t)/R(t). \tag{27}$$

The *price formation mechanism* (Tarasyev and Zhu, 2012) for exhausted resources has the form of the proportion between the current price $p(t)$ to the initial price p_0, and the initial level M_0 to the current resource stock ($M_0 - M(t)$), and is presented by the relation

$$\frac{p(t)}{p_0} = \frac{M_0}{M_0 - M(t)}, \quad p(t) = \frac{p_0 M_0}{M_0 - M(t)}. \tag{28}$$

Under the condition on closedness of the economic system, the following *balance equality* takes place

$$y(t) = c(t) + R(t)p(t) + u_K(t)y(t) + u_Z(t)y(t). \tag{29}$$

This expression presumes that the output $y(t)$ can be spent on consumption $c(t)$, current expenses on natural resources $R(t)p(t)$, investments in capital $u_K(t)y(t)$, and investments in increasing the resource productivity $u_Z(t)y(t)$. The balance relation imposes restrictions on the investment components of the model: $u_K(t) \geqslant 0$, $u_Z(t) \geqslant 0$, and $0 \leqslant u_K(t)+u_Z(t) \leqslant 1-\dfrac{c(t)}{y(t)}-\dfrac{p(t)R(t)}{y(t)} < 1$. Using obtained inequalities one can determine positive constants $\overline{u}_K$ and $\overline{u}_Z$, $(\overline{u}_K + \overline{u}_Z) < 1$, s.t. investment components belong to a compactum for all t, $t \in (t_0, +\infty)$

$$(u_K(t),\, u_Z(t)) \in [0,\, \overline{u}_K,] \times [0,\, \overline{u}_Z,] = \mathbb{U}_K \times \mathbb{U}_Z = \mathbb{U}. \tag{30}$$

The consumption level $c(t)$ is derived from the balance equation (29)

$$\begin{aligned} c(t) &= y(t)\left(1 - p(t)R(t)/y(t) - u_K(t) - u_Z(t)\right) \approx \\ &\approx y(t)\left(1 - p(t)R(t)/y(t) - u_Z(t)\right)\left(1 - u_K(t)\right) \end{aligned} \tag{31}$$

Remark 7. The formula (31) for the consumption $c(t)$ ensures the positiveness of its values. Consequently, there exists a positive constant P. s.t.

$$0 \leqslant p(t)R(t)/y(t) \leqslant P < 1 - \overline{u}, \quad \forall\, t \geqslant 0. \tag{32}$$

It is worth to mention that the fraction $p(t)R(t)/y(t)$ presents the share of output spent on natural resources.

Model dynamics

We assume that dynamics of the capital stock $K(t)$ satisfy equations of the Solow model (Tarasyev and Zhu, 2012)

$$\dot{K}(t) = u_K(t)y(t) - \mu K(t), \quad K(0) = K_0, \tag{33}$$

where a positive parameter μ denotes the capital depreciation rate. Relative growth of the resource productivity $Z(t)$ is supposed to be proportional to a portion of assigned investments $u_Z(t)$

$$\dot{Z}(t) = \beta u_Z(t) Z(t), \quad Z(0) = Z_0. \tag{34}$$

A nonnegative parameter β describes the effectiveness of investments $u_Z(t)$ in growth of the resource productivity. Using definitions of the resource productivity (27), the production function (26), and the capital dynamics (33), one can derive the following expression for the dynamics of resources consumption

$$\frac{\dot{R}(t)}{R(t)} = \frac{b - \beta u_Z(t)}{1-\alpha} + u_K(t)\frac{y(t)}{K(t)} - \mu, \quad R(0) = R_0. \tag{35}$$

For further analysis of the model, it is convenient to introduce new phase variables $x(t) = (x_1(t),\, x_2(t),\, x_3(t))$

$$x_1(t) = e^{\frac{b}{\alpha}t}\frac{p_0}{p(t)},\; x_2(t) = e^{\frac{b}{\alpha}t}R(t),\; x_3(t) = \frac{y(t)}{K(t)}. \tag{36}$$

Taking into account formulas (33), (34), relations for the price formation mechanism (28), and cumulative resource consumption (25), one can derive the dynamics for the introduced variables

$$\dot{x} = \Phi(x,u) = F(x)u + G(x), \quad \text{where} \tag{37}$$

$$F(x) = \frac{\beta}{1-\alpha}\begin{pmatrix} 0 & 0 \\ -x_2 & (1-\alpha)x_2x_3 \\ -\alpha x_3 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} u_Z \\ u_K \end{pmatrix},$$

$$G(x) = \left(\frac{\beta}{\alpha}x_1 - \frac{1}{M_0}x_2;\; \left(\frac{b}{\alpha(1-\alpha)} - \mu\right)x_2;\; \frac{b}{1-\alpha}x_3\right)^T$$

Initial conditions x_0 for the state vector x are

$$x(0) = x_0 = (x_1^0,\, x_2^0,\, x_3^0)^T = \left(1,\, R_0,\, a\,(R_0/K_0)^{\alpha}\right)^T \tag{38}$$

The consumption level $c(t) = c(x(t), u(t))$ (31) in new variables has the form

$$\begin{aligned} &c(x,u) = y(x)(1-u_K)\left(1 - u_Z - \vartheta q_1(x)\right), \\ &y(x) = a^{\frac{1}{\alpha}}x_2 x_3^{1-\frac{1}{\alpha}},\; q_1(x) = \frac{p_0}{\vartheta a^{\frac{1}{\alpha}}}\frac{x_3^{\frac{1}{\alpha}-1}}{x_1},\; \vartheta = \frac{1-\alpha}{\beta}. \end{aligned} \tag{39}$$

Optimal control problem

Problem 8. The optimal control problem is to maximize the *utility function* with the positive discount factor ρ

$$J = \int_0^{+\infty} e^{-\rho t}\ln c(t)dt = \int_0^{+\infty} e^{-\rho t}W(x(t),u(t))dt \tag{40}$$

over control processes $(x(t), u(t))$ of the system (37), satisfying constraints (30) for vector $u(t) = (u_K(t), u_Z(t))^T$.

Let us note that the posed problem is an optimal control problem with infinite horizon. For its solution, one can use generalizations of the Pontryagin maximum principle obtained in (Aseev and Kryazhimskiy, 2007; Pontryagin et al., 1962). One can also refer to applications of methods of the optimal control theory for analysis of economical growth models presented in the papers (Arrow, 1985; Balder, 1983; Grass et al., 2008), and also in the works of authors (Tarasyev and Zhu, 2012; Tarasyev and Usova, 2016; Krasovskii et al., 2008).

Remark 9. Numerical results presented below are derived for the following values of the model parameters $a = 1.0$, $b = 0.015$, $\alpha = 0.39 < 0.5$, $\beta = 1.523$, $\mu = 0.2$, $M_0 = 50$, $\overline{u}_Z = 0.3$, $\overline{u}_K = 0.6$, $R_0 = 0.2105$, $K_0 = 26.6$, $p_0 = 3.06$. These parameters are statistically calibrated Tarasyev and Zhu (2012); Tarasyev and Usova (2016) using econometric methods. To illustrate the robustness of the proposed stabilization technique, the discount factor ρ is not fixed.

5.2 Optimal control problem analysis

Hamiltonian function of the Optimal Control Problem

According to the generalized Pontryagin maximum principle (Aseev and Kryazhimskiy, 2007) the stationary Hamiltonian function of Problem 8 is determined as follows

$$\widetilde{H}(x,u,\psi) = W(x,u) + \psi^T\Phi(x,u), \tag{41}$$

where $\psi = \psi(t)$ denotes a vector of adjoint variables.

Function $W(x,u)$ is strictly concave in control variables, since its Hessian matrix has a diagonal form $\partial^2 W_{u^2} = \mathrm{diag}\left\{-(1-u_K)^{-2};\quad -(1-\vartheta q_1(x) - u_Z)^{-2}\right\}$.

The strict concavity of function $W(x,u)$ and affinity of the right-hand side $\Phi(x,u)$ of dynamics (37) with respect to control parameters $u = (u_K, u_Z)$ imply that the Hamiltonian function (41) is a strictly concave function of control variables $u = (u_K, u_Z)^T$. Using formula (9), control parameters $u^0(x,\psi) = (u_K^0, u_Z^0) \in \mathbb{U}$ maximizing the Hamiltonian function $\widetilde{H}$, can be presented in the form

$$u_K^0 = \begin{cases} 0, & u_K^* < 0 \\ u_K^*, & u_K^* \in \mathbb{U}_K, \\ \overline{u}_K, & u_K^* > \overline{u}_K \end{cases} \quad u_Z^0 = \begin{cases} 0, & u_Z^* < 0 \\ u_Z^*, & u_Z^* \in \mathbb{U}_Z \\ \overline{u}_Z, & u_Z^* > \overline{u}_Z. \end{cases} \quad (42)$$

Let $q_2(x,\psi) := -x_2\psi_2 - \alpha x_3\psi_3$. The transition control regime (u_K^*, u_Z^*) satisfies formulas $u_K^* = 1 - (x_2x_3\psi_2)^{-1}$, $u_Z^* = 1 - \vartheta\left(q_1(x) + q_2^{-1}(x,\psi)\right)$.

The maximized Hamiltonian function is determined by the formula $H(x,\psi) = \max\limits_{u\in\mathbb{U}} \widetilde{H}(x,\psi,u) = \widetilde{H}(x,\psi,\widehat{u})$.

Hamiltonian System

The Hamiltonian system has the form

$\dot{x}(t) = H'_\psi(x(t),\psi(t)), \qquad \dot{\psi}(t) = \rho\psi(t) - H'_x(x(t),\psi(t))$.

In the domain $\mathcal{D}$ of the transition control regime

$$\mathcal{D} = \left\{ (x,\psi): \begin{array}{l} 0 \leqslant 1 - \vartheta(q_1(x) + q_2^{-1}(x,\psi)) \leqslant \overline{u}_Z \\ \text{and } 1 - \overline{u}_K \leqslant x_2x_3\psi_2 \leqslant 1 \end{array} \right\}$$

the Hamiltonian system has the form

$$\begin{aligned} &\dot{x}_1 = (b/\alpha)\,x_1 - M_0^{-1}x_2;\ \gamma = (\alpha\beta - b)/(\alpha(1-\alpha)) + \mu \\ &\dot{x}_2 = x_2\left(q_1(x) + q_2^{-1}(x,\psi) + x_3 - \gamma\right), \\ &\dot{x}_3 = \alpha\left(q_1(x) + q_2^{-1}(x,\psi)\right) + \mu - \gamma, \\ &\dot{\psi}_1 = (\rho - b/\alpha)\,\psi_1 - q_1(x)q_2(x,\psi)/x_1, \\ &\dot{\psi}_2 = (\rho + \gamma - x_3 - q_1(x) - q_2^{-1}(x,\psi))\ psi_2 + \psi_1/M_0, \\ &\dot{\psi}_3 = -(\gamma - \mu - \rho)\,\psi_3 + q_1(x)x_2\psi_2/x_3 - \\ &\qquad - x_2\psi_2/(x_3q_2(x,\psi)) - x_2\ psi_2 + (\alpha x_3)^{-1}, \end{aligned} \quad (43)$$

System (43) has a steady state $P^* = (x^*,\psi^*) \in \mathcal{D}$ satisfying conditions A1 and A2, if the equality holds

$$b/\alpha < \rho < \beta - b/\alpha. \quad (44)$$

We want to show here that for almost all values of parameters ρ in (44), there exists a stable manifold Ω, constructed by the proposed algorithm.

Remark 10. The existence of a steady state with positive phase coordinates means that the optimal trajectory converging to the steady state (Hartman, 1964) has the sustainability property (zero values of a steady state provide collapse in economic development), even in the case of exhausting resources due to the effect of substitution for natural resources in production factors.

First of all, let us analyze the characteristic polynomial of the matrix M (17). Matrix M is a real Hamiltonian matrix, and its characteristic polynomial is an even function of μ

$$\mu^6 + a_2\mu^4 + a_4\mu^2 + a_6 = 0. \quad (45)$$

Introducing variables $\widetilde{\mu} = \mu^2$, we get the cubic polynomial

$$\widetilde{\mu}^3 + a_2\widetilde{\mu}^2 + a_4\widetilde{\mu} + a_6 = 0. \quad (46)$$

Roots (46) can be analyzed using Cardano method

(1) Compute values of Q_1 and Q_2 by the formulae $Q_1 = a_4 - 3\,(a_2/3)^2$, $Q_2 = 2\,(a_2/3)^3 - a_4\,(a_2/3) + a_6$.
(2) Find the discriminant: $Q = (Q_1/3)^3 + (Q_2/2)^2$.
(3) If $Q < 0$ then equation (46) has 3 real eigenvalues; if $Q > 0$ then equation (46) has one real and two complex-conjugate roots; and if $Q = 0$ then roots are real and at least two of them coincide.

In the considered model, we calculate Q (see Fig. 1) as a function of parameter ρ belonging to (44) One can see, that for $\rho < \rho_0 = 0.096$ we have three real roots, and for $\rho > \rho_0$, the polynomial has one real and two complex-conjugate

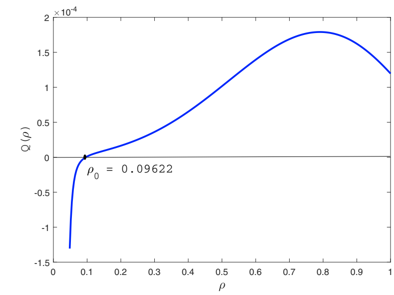

Fig. 1. Discriminant of the cubic polynomial, $Q = Q(\rho)$.

roots (here we exclude the value $\rho = \rho_0$). Nevertheless, in both cases we have exactly three eigenvalues with negative real parts, which are solutions of (45) and $\mu_i^- = -\sqrt{\widetilde{\mu}_i}$, where $\widetilde{\mu}_i$ are roots of (46). It implies that the stable manifold Ω (19) exists.

Let us construct stable manifolds for the cases $\rho_s = 0.09 < \rho_0$ and $\rho_f = 0.18 > \rho_0$. The chosen parameters ρ_s and ρ_f belong to the interval (44), thus, for both parameters we can find coordinates of the steady state

$$\rho_s: \begin{cases} x_1^* = 0.9681, & x_2^* = 1.8629, & x_3^* = 0.3768, \\ \psi_1^* = 13.1772, & \psi_2^* = 3.0362, & \psi_3^* = -43.1543. \end{cases}$$
$$\rho_f: \begin{cases} x_1^* = 1.0102, & x_2^* = 1.9439, & x_3^* = 0.4444, \\ \psi_1^* = 6.7751, & \psi_2^* = 2.1052, & \psi_3^* = -28.3021. \end{cases}$$

For both values of parameter ρ, matrices $M = M(\rho)$ have exactly three eigenvalues with negative real parts

$$\begin{aligned} \rho_s:&\quad \mu_1^- = -1.1173,\ \mu_2^- = -0.1331,\ \mu_3^- = -0.0906 \\ \rho_f:&\quad \mu_1^- = -1.0117,\ \mu_2^- = -0.1481 + 0.0636i = \overline{\mu_3^-}. \end{aligned}$$

Moreover, real parts of these eigenvalues are less than $-\rho_i/2$, $i = \{s,f\}$, and it implies that the linearized Hamiltonian system (14) constructed for the resource productivity model, has asymptotically stable solutions. In the case $\rho = \rho_s$, we get real eigenvalues and, as a result, the saddle character of the steady state. In the second case $\rho = \rho_f$, the steady state is focal. For the obtained eigenvalues, we pick the corresponding eigenvectors V_i (20) and construct solution X_i of the problem 3 ($i = \{s,f\}$)

$$X_s = V_{s_2}V_{s_1}^{-1} = \begin{pmatrix} -17.145 & -0.819 & 19.344 \\ -0.819 & -1.727 & -0.224 \\ 19.344 & -0.224 & 96.646 \end{pmatrix} \quad (47)$$

$$X_f = V_{f_2}V_{f_1}^{-1} = \begin{pmatrix} -0.452 & -0.230 & -0.315 \\ 0.500 & 0.250 & 0.322 \\ -1.003 & -0.646 & -0.699 \end{pmatrix} \quad (48)$$

Hence, the conjugate variables ψ can be expressed through the phase vector x, using formula (15) and matrix X_i

$$\widehat{\psi}_i(x) = \psi_i^* + X_i(x - x_i^*), \quad i = \{s,f\}.$$

The nonlinear stabilizer for the Hamiltonian dynamics (43) is found by the formulae $\widehat{u}_K^{(i)} = 1 - \left(x_2x_3\widehat{\psi}_{i_2}(x)\right)^{-1}$ and $\widehat{u}_Z^{(i)} = 1 - \vartheta\left(q_1(x) + \left(q_2(x,\widehat{\psi}_i(x))\right)^{-1}\right)$.

The stabilized Hamiltonian system is derived from the system (37) by substituting the obtained nonlinear stabilizer $\widehat{u}(x) = (\widehat{u}_K(x), \widehat{u}_Z(x))$ instead of controls u_K, u_Z.

$$\begin{aligned} &\dot{x}_1 = bx_1/\alpha - x_2/M_0 \\ &\dot{x}_2 = x_2\left(q_1(x) + q_2^{-1}(x,\widehat{\psi}_i(x)) - \left(x_2\widehat{\psi}_{i_2}(x)\right)^{-1} + x_3 - \gamma\right) \\ &\dot{x}_3 = \alpha x_3\left(q_1(x) + q_2^{-1}(x,\widehat{\psi}_i(x)) - \left(x_2\widehat{\psi}_{i_2}(x)\right)^{-1} - \gamma + \mu\right) \end{aligned} \quad (49)$$

The derived *stabilized system* is a closed loop system in phase variables $x(t) = (x_1(t), x_2(t), x_3(t))$.

Remark 11. Solutions of the stabilized Hamiltonian system (49) provide a good approximation of optimal trajectories at least in a neighborhood of the steady state P^*, since they have similar qualitative behaviors, specifically, both trajectories converge to the steady state and have equal tangential slopes. From this point of view, solutions of the stabilized Hamiltonian system (49) can be called *suboptimal stabilized trajectories*, since the stabilized trajectory approximates the optimal trajectory with *quadratic precision* in a small vicinity of the steady state.

Phase portraits of solutions of the stabilized Hamiltonian system (49) constructed in the steady state neighborhood with radius $\delta = 0.01$ are presented on Fig. 2, 3.

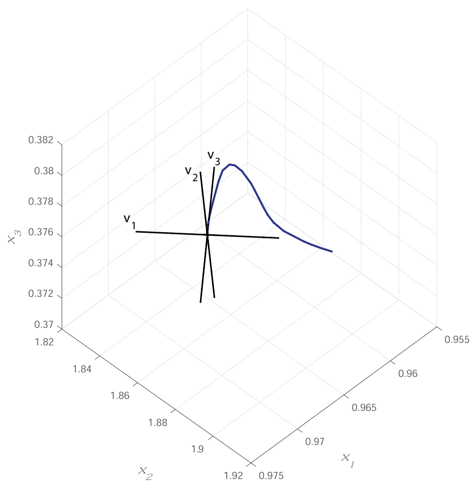

Fig. 2. Phase trajectory $x_3 = x_3(x_1, x_2)$ for $\rho = \rho_s$.

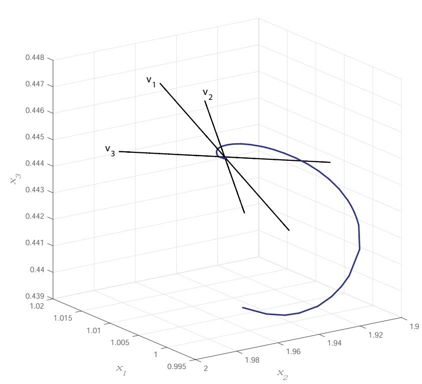

Fig. 3. Phase trajectory $x_3 = x_3(x_1, x_2)$ for $\rho = \rho_f$.

In the first case, the phase trajectory slides to the saddle steady state by the vector v_3 corresponding to the maximal negative eigenvalue, while in the second case, solution has the spiral shape in the plane generated by eigenvectors v_2 and v_3 corresponding to the complex–conjugate pairs of eigenvalues with negative real parts. This spiral spools on the eigenvector v_1 (corresponding to the negative real eigenvalue) and approaches the steady state.

6. CONCLUSION

The paper is devoted to development of the robust method for stabilization of the Hamiltonian systems arising in economic growth models. Due to the structure of the Jacobian evaluated at the steady state, the suggested approach for system stabilization is robust with respect to the model parameters. The only requirement posed on parameters is the existence of a steady state.

Application of the proposed technique to the resource productivity model demonstrates the efficiency of the proposed approach. Particularly, in this model the stable manifold exists for focal and saddle steady states. The obtained results imply that for almost all parameters ρ satisfying the existence condition for steady states (44), the Hamiltonian system (43) is stabilizable.

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